ON THE THEORY OF DIFFRACTION OF WEAK SHOCK WAVES ROUND CONTOURS OF ARBITRARY SHAPE

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This paper considers the classical problem of diffraction of a plane weak shock wave at contours of arbitrary shape. A different approach is here offered to the formulation and solution of such problems. The essence of the method consists in the fact that the plane unsteady problem of diffraction is reduced to an auxiliary three-dimensional steady problem, where the third coordinate is proportional to the time.

In the solution of diffraction problems the space-time system of coordinates has indeed been applied before. For example, [1] solved in this way the problem of diffraction of a weak plane shock wave by thin profiles of arbitrary shape, when the front of the incident shock wave is propagated along the chord of the profile. But the methods considered in the present paper and in [1] are essentially different.

In the present paper problems of diffraction are solved in a linear formulation which, generally speaking, is permissible over only a part of the region of diffraction. Khristianovich and his pupils showed in [2] that when the front of the incident shock wave makes a small angle with the normal to the contour, the reflection of the weak shock becomes a Mach reflection - not a regular reflection. Accordingly in the neighborhood of the Mach reflection the solution of the linear problem gives a qualitatively false picture of diffraction.

In our paper the effect mentioned will also occur at certain points of the contour in the first moments of time. But we shall linearise the problem in the whole region of the diffraction field for the reason that the linearisation of the basic equations of the problem is invalid only in small regions of the diffraction field and for very short durations of time. Accordingly, the introduction of nonlinear terms in these small regions greatly complicates the already complicated mathematical formulation of the problem, whilst it changes very little the general pattern of diffraction for arbitrary durations of time.

In the first sections the contours are assumed to be rigid. In the later parts the method is extended to deal also with deformable contours. In the case of a flat plate it is shown that the problem of diffraction can also be solved to the second approximation by the given method.

1. Formulation of the general problem of diffraction of shock waves by nondeformable contours. 1. Suppose that a weak steady shock wave impinges on a closed contour C of arbitrary shape in the xy-plane (Fig. 1). We shall assume that the flow past the contour C by the shock wave is isentropic and irrotational; viscosity and heat conduction are neglected.

For the given assumptions the problem of diffraction of a shock wave round a nondeformable contour C reduces to the solution of the wave equation for the potential Φ

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=\frac{\partial^2\Phi}{\partial\tau^2}, \qquad \tau=\frac{at}{l}, \qquad x=\frac{x'}{l}, \qquad y=\frac{y'}{l} \tag{1.1}$$

with the following boundary conditions on C and on the front ABD of the reflected shock wave

$$\frac{\partial\Phi}{\partial n}=0 \quad \text{on } C, \qquad \Phi(x, y, \tau)=\Phi_0(x, y, \tau) \quad \text{on } ABD$$
$$\Phi(x, y, \tau)=\Phi_0(x, y, \tau) \quad \text{when } \tau\leqslant 0 \tag{1.2}$$

Here x, y, τ are dimensionless coordinates and time respectively; Φ is the dimensionless velocity potential; a is the speed of sound; l is a characteristic length of the problem; n is the exterior normal to the contour; Φ_0 is the dimensionless potential of the stream beyond the front of the incident shock wave.

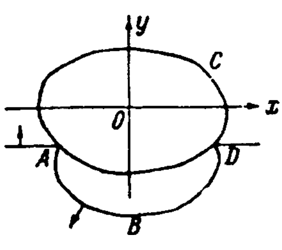

Fig. 1.

The commencement of diffraction of the shock wave by the contour C is taken as the instant when $\tau = 0$. For simplicity we shall assume that the flow parameters of the gas behind the front of the incident shock wave are constant, i.e.

$$\Phi_0(x, y, \tau)=(\Delta p/\rho a^2)(y-\tau) \tag{1.3}$$

Here Δp is the pressure difference across the front of the incident shock wave; ρ is the gas density. When $\tau > 0$ the potential Φ will be sought in the form

$$\Phi(x, y, \tau) = \Phi_0(x, y, \tau) + \varphi(x, y, \tau) \tag{1.4}$$

It is not diffuclt to see that the perturbation velocity potential $\varphi(x, y, \tau)$ satisfies the wave equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{\partial^2\varphi}{\partial \tau^2} \tag{1.5}$$

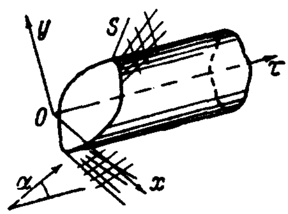

Fig. 2.

in the region between the reflected shock wave ABD and the contour C and the conditions

$$\frac{\partial\varphi}{\partial n} = -\alpha_0 \frac{\partial y}{\partial n} \quad \text{on } C$$

$$\varphi = 0 \quad \text{when } \tau \leqslant 0, \qquad \varphi = 0 \quad \text{on } ABD \tag{1.6}$$

2. Let us consider the auxiliary problem of flow of a three-dimensional steady supersonic stream of ideal gas past a semi-infinite hollow cylinder (Fig. 2) with generators parallel to the axis of τ, at a small angle of attack α. Suppose that the curve, generated by the cross-section of the given hollow cylinder by an arbitrary plane τ = const, corresponds to that part of the contour C, forming part of the boundary of the perturbed region of flow of the incident shock wave past the contour C (Fig. 1), or else it corresponds to the whole of the contour C when $\tau \geqslant \tau_2$, where τ_2 is the time at which the incident shock wave has completely engulfed the contour C.

The potential $\varphi(x, y, \tau)$ for flow past the hollow cylinder satisfies the equation

$$(M^2 - 1)\frac{\partial^2\varphi}{\partial \tau^2} = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} \tag{1.7}$$

and the conditions

$$\frac{\partial\varphi}{\partial n} = -\alpha \frac{\partial y}{\partial n} \quad \text{on the surface of the cylinder}$$

$$\varphi = 0 \quad \text{when } \tau = 0, \qquad \varphi = 0 \quad \text{on the wave surface} \tag{1.8}$$

In the given auxiliary problem we shall consider only the exterior problem, whilst the perturbations from the internal surface will not be taken account of, which is permissible by virtue of the linearity of the equations (1.7), (1.8).

Let us compare the system of equations (1.5), (1.6) with the system (1.7), (1.8).

It is not difficult to see that the equations (1.5) and (1.7) coincide for the value $M = \sqrt{2}$. The conditions (1.6) and (1.8), except the last, coincide when $\alpha = \alpha_0$.

Let us prove that the condition $\varphi = 0$ at the front of the reflected shock wave in (1.6) and $\varphi = 0$ on the wave surface in (1.8) are equivalent.

For this purpose let us consider the pattern of diffraction in the problem (1.5), (1.6) at the instant of time $\tau = \tau_1$. When $\tau = \tau_1 (\tau_1 < \tau_2)$ the boundary of the perturbed region consists of part of the contour C (or the whole of the contour C when $\tau_1 > \tau_2$) and the front of the reflected shock wave. In its turn the cross-section of the cylinder by the plane $\tau = \tau_1$ in the auxiliary problem corresponds to the contour C in problem (1.5) to (1.7). But since the wave surface in the auxiliary exterior problem turns out to be the envelope of the characteristic cones arising from the points of the leading edge of the hollow cylinder with a semi-angle of $\pi/4$, then it is not difficult to see that the section curve of the given wave surface by the plane $\tau = \tau_1$ also corresponds to the curve forming the front of the reflected shock wave in problem (1.5), (1.6). Consequently, the last of the conditions in (1.6) and (1.8), respectively, also coincide.

Accordingly the system (1.5), (1.6) completely coincides with the system (1.7), (1.8) and the solution of the auxiliary problem for $\alpha = \alpha_0$ and $M = \sqrt{2}$ will give the solution of the problem of diffraction of a weak shock wave by the contour C. Let us consider particular cases, when the contour C is completely defined.

2. Diffraction of a shock wave by a circle. Let us pass from rectangular coordinates x, y to polar coordinates r, θ, in which the system (1.5), (1.6) takes the form

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = \frac{\partial^2\varphi}{\partial\tau^2} \qquad (2.1)$$

$$\frac{\partial\varphi}{\partial r} = -\alpha_0 \sin\theta \quad \text{when } r = 1$$

$$\varphi = 0 \quad \text{when } \tau = 0 \qquad (2.2)$$

$\varphi = 0$ at the front of the reflected shock wave (2.3)

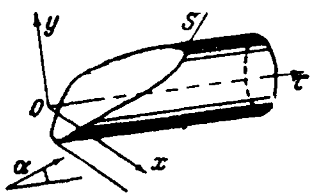

Fig. 3.

By virtue of Section 1, the given problem is equivalent to the exterior problem of flow past a truncated semi-infinite hollow circular cylinder (Fig. 3) by a three-dimensional steady supersonic stream of ideal gas with $M = \sqrt{2}$ at a small angle of attack $\alpha_0 = \Delta p/\rho a^2$.

For the solution of the system (2.1) to (2.3), describing also the solution of the auxiliary problem, let us use Volterra's method [3].

Then when $\tau > 0$ the solution of the auxiliary problem reduces to determination of the potential $\varphi(1, \theta, \tau)$ on the surface of the appropriate cylinder $r = 1$, satisfying the following integral equation of Volterra's type II:

$$\varphi(1, \theta_0, \tau_0) = \frac{1}{2\pi} \frac{\partial}{\partial \tau_0} \left\{ \iint_{\Sigma} \left[\varphi(1, \theta, \tau) \frac{\partial V}{\partial r} - \frac{\partial \varphi}{\partial r} V \right] d\sigma \right\} \tag{2.4}$$

Here Volterra's function is given by

$$V = \ln \frac{(\tau_0 - \tau) + \sqrt{(\tau_0 - \tau)^2 - (r_0 - r)^2 - 4r_0 r \sin^2[(\theta_0 - \theta)/2]}}{\sqrt{(r_0 - r)^2 + 4r_0 r \sin^2[(\theta_0 - \theta)/2]}}$$

The region of integration Σ is the part of the surface of the cylinder (Fig. 3) cut off by the cone of influence from the point ($r_0 = 1$, θ_0, τ_0).

By virtue of the equivalence of the problems the potential $\varphi(1, \theta, \tau)$ on the contour C (Fig. 1) for any value of τ will also satisfy the integral equation (2.4).

Equation (2.4) can always be solved, e.g. numerically. For large τ we can construct the asymptotic solution of the problem (2.1) to (2.3).

By the same token, for large τ the radius r of the wave surface (Fig. 3) depends weakly on θ and then the function $\varphi(x, y, \tau)$ can be sought in the form

$$\varphi(x, y, \tau) = -x_0 \sin\theta\, f(r, \tau) \tag{2.5}$$

Let us substitute (2.5) in (2.1) to (2.3) and to the new system so obtained let us apply Laplace's transformation. Then we have

$$f(r, \tau) = \frac{1}{2\pi i} \int_{M} \frac{K_1(r, q)}{K_1(q)\, q^2} e^{q\tau} dq \tag{2.6}$$

where $K_1(q)$ is the Bessel function of imaginary argument; M is the contour in the plane of q, along which the integration proceeds in the application of the inverse Laplace transformation. Determining $\varphi(r, \theta, \tau)$ from equations (2.4) or (2.5) for large τ we can find, for example, the pressure distribution p along C for any value of τ

$$p(1, \theta, \tau) = p_0 - \rho a^2 \left(\frac{\partial \varphi}{\partial \tau} \right)_{r=1} \tag{2.7}$$

For large values of τ we have [4]

$$p(1, \theta, \tau) = p_0 - \rho a^2 \sin\theta \left\{ e^{-0.6135\tau} [1.212 \cos(0.5012\tau) + \right. \tag{2.8}$$

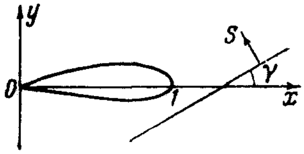

Fig. 4.

$$+ 0.1898 \sin (0.5012\tau)] - \int_0^\infty \frac{e^{-q\tau}}{K_1'^2(q) + \pi^2 I_1'^2(q)} \frac{dq}{q^2}\}$$

where $I_1(q)$ is the Bessel function with imaginary argument.

From the form of the solution (2.6) and (2.8) it follows that the reflected wave behaves as an outgoing wave, as indeed it must from physical considerations.

3. The case when the contour C is a thin profile. We shall assume that a thin profile C moves (Fig. 4) with a certain velocity U at zero angle of attack along the x-axis and that the shock wave travels at a certain angle γ to the x-axis.

Moreover, we shall solve the weakly nonlinear problem, i.e. we shall take account also of terms of the second order of smallness in the equation for the perturbation velocity potential $\varphi(x, y, \tau)$, and the interaction of the shock wave with perturbations arising from the moving profile will be ignored.

Assuming that the motion of the shock wave past the profile C is irrotational and isentropic (k is the adiabatic index), we reduce the problem to determination of the perturbation velocity potential $\varphi(x,y,\tau)$ satisfying the equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - \frac{\partial^2\varphi}{\partial \tau^2} = (k-1)\frac{\partial\varphi}{\partial\tau}\frac{\partial^2\varphi}{\partial\tau^2} + 2\left(\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x\partial\tau} + \frac{\partial\varphi}{\partial y}\frac{\partial^2\varphi}{\partial y\partial\tau}\right) \tag{3.1}$$

with accuracy up to terms of the third order of smallness with respect to $\varphi(x, y, \tau)$ and the conditions

$$\partial\varphi / \partial y = -\alpha_0 \cos\gamma + (-\alpha_0 \sin\gamma + \partial\varphi / \partial x) f'(x) \quad \text{on } C \tag{3.2}$$

$$\varphi = 0 \quad \text{when } \tau = 0, \qquad \varphi = 0 \quad \text{at the front of the reflected wave} \tag{3.3}$$

$$(\partial\varphi / \partial\tau)_{y\to-0} = (\partial\varphi / \partial\tau)_{y\to+0} \quad \text{when } u < a \tag{3.4}$$

In this condition of Zhukovskii on the trailing edge and the vortex sheet, $y = f(x)$ is the equation of the profile C. Let us set $\varphi(x, y, \tau) = \varphi_1(x, y, \tau) + \varphi_2(x, y, \tau) + \ldots$; then φ_1 and φ_2 satisfy the equations

$$\frac{\partial^2\varphi_1}{\partial x^2} + \frac{\partial^2\varphi_1}{\partial y^2} = \frac{\partial^2\varphi_1}{\partial \tau^2} \tag{3.5}$$

$$\frac{\partial^2\varphi_2}{\partial x^2} + \frac{\partial^2\varphi_2}{\partial y^2} - \frac{\partial^2\varphi_2}{\partial \tau^2} = (k-1)\frac{\partial\varphi_1}{\partial\tau}\frac{\partial^2\varphi_1}{\partial\tau^2} + 2\left(\frac{\partial\varphi_1}{\partial x}\frac{\partial^2\varphi_1}{\partial x\partial\tau} + \frac{\partial\varphi_1}{\partial y}\frac{\partial^2\varphi_1}{\partial y\partial\tau}\right) \tag{3.6}$$

and the conditions

$$\varphi_1 = \varphi_2 = 0 \quad \text{when } \tau = 0, \qquad \varphi_1 = \varphi_2 = 0 \tag{3.7}$$

outside the reflected shock wave

$$\frac{\partial \varphi_1}{\partial y} = -\alpha_0 \cos\gamma, \qquad \frac{\partial \varphi_2}{\partial y} = \left(-\alpha_0 \sin\gamma + \frac{\partial \varphi_1}{\partial x}\right) f'(x) \quad \text{when } y = 0 \tag{3.8}$$

$$\left(\frac{\partial \varphi_i}{\partial \tau}\right)_{y \to -0} = \left(\frac{\partial \varphi_i}{\partial \tau}\right)_{y \to +0} \quad \text{when } u < a \tag{3.9}$$

Following the method of Section 1, it can be shown that the plane problem of flow of a weak shock wave past a moving thin profile is equivalent to the problem of flow of a steady supersonic three-dimensional stream of ideal gas with $M = \sqrt{2}$ past a corresponding thin wing at a small angle of attack $\alpha_0 \cos \gamma$ in two cases.

1) Flow past a moving flat plate

$$f'(x) = 0, \qquad 2k_1 = k - 1, \qquad M_0 = \frac{u}{a} \neq 0 \qquad (\gamma \text{ — is an arbitrary angle})$$

2) Flow past a fixed thin profile of a shock wave from exactly below

$$f'(x) \neq 0, \qquad M_0 = 0, \qquad 2k_1 = k - 1, \qquad \gamma = 0$$

(k_1 — is the adiabatic index in the auxiliary problem)

The solution of the linear problem for a flat plate with $M_0 > 1$ was given by Golubinskii [5].

Let us derive the solution of the problem of flow of a weak shock wave with $\gamma = 0$ past a flat plate moving with $M_0 > 1$ up to the second approximation, assuming that $\varphi_1(x, y, \tau)$ is known [5].

Since the given problem is equivalent to the flow of a three-dimensional steady supersonic stream with $M = \sqrt{2}$ past the appropriate airfoil (in the form of a semi-infinite flat plate) (Fig. 5), then we shall solve the appropriate auxiliary problem. The equations of the side edges of the airfoil (Fig. 5) have the form

$$x = M_0 \tau \, (AC), \qquad x = 1 + M_0 \tau \, (BD) \tag{3.10}$$

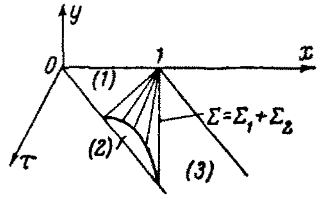

Fig. 5.

The whole surface of the plate can be divided into three regions, as shown in Fig. 5. The function $\varphi_2(x, y, \tau)$ will be determined in each of the given regions. Then to the conditions (3.7), (3.8) we have to add the condition of continuity of potential on transition through the surface Σ. On one part of the surface Σ (let us denote it by Σ_1) the solution in the region (2) will be joined to the solution in region (1), whilst on the other, Σ_2, it is joined to the solution in region (3). The line dividing the surfaces Σ_1 and Σ_2 will correspond to the line of zero potential, since it is a generator in fact for the whole wave surface of the flat plate, for which the potential is taken as equal to zero.

Let us denote the solutions in the regions (1) and (3) by φ_{21} and φ_{23} respectively. It is easy to see that the solutions in the regions (1) and (3) correspond to the solutions of flow past a flat plate without sweep and with sweep, respectively, and then φ_{21} and φ_{23} have the form

$$\varphi_{21}(x, y, \tau) = \alpha_0 [\tau - (2\alpha_0 + 1) y] - \alpha_0^2 (k_1\tau + y) \tag{3.11}$$

$$\varphi_{23}(x, y, \tau) = \frac{\alpha_0}{\sqrt{\cos 2\chi}} [(\tau - 2\alpha_0 y)\cos\chi - x\sin\chi - \sqrt{\cos 2\chi}\, y] - \qquad \left(\tan\chi = \frac{1}{M_0}\right)$$
$$- \frac{2\alpha_0^2}{\cos 2\chi}\left[\frac{(k_1 + 1)\cos^2\chi}{2\cos 2\chi}(\tau\cos\chi - x\sin\chi) - \frac{(\tau\cos\chi - x\sin\chi)}{2\cos^2\chi} - \sqrt{\cos 2\chi}\, y\right] \tag{3.12}$$

For solution of equation (3.6) in the region (2) with the boundary conditions (3.7), (3.8) and the condition of continuity of potential at the surface Σ we apply the modified method of Volterra. Then for $\varphi_{22}(x, y, \tau)$ we obtain

$$\varphi_{22}(x, y, \tau) = \frac{1}{\pi}\frac{\partial}{\partial\tau_0}\left\{\iint\limits_{\Sigma_1}\left(\varphi_{21}\frac{\partial V}{\partial N} - V\frac{\partial\varphi_{21}}{\partial N}\right)d\sigma + \right.$$
$$\left. + \iint\limits_{\Sigma_2}\left(\varphi_{23}\frac{\partial V}{\partial N} - V\frac{\partial\varphi_{23}}{\partial N}\right)d\sigma + \iiint\limits_{Q} VF(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta\right\} \tag{3.13}$$

Here Q is the volume bounded by the plane $y = 0$, the surface of the cone of influence at the point $(x, 0, \tau)$ and the part of the surface Σ cut off by the given cone of influence; $F(\tau, x, y)$ is the right-hand side of equation (3.6); N is the conormal to the surface Σ; Volterra's function has the form

$$V = \ln\frac{(\tau - \varepsilon) + \sqrt{(\tau - \varepsilon)^2 - (x - \eta)^2 - (y - \zeta)^2}}{\sqrt{(x - \eta)^2 + (y - \zeta)^2}} \tag{3.14}$$

Since the surface Σ is characteristic, the conormal N lies on the surface Σ and $\partial/\partial N = \partial/\partial s$, where $s = \sqrt{(x^2 + y^2)}/\tau$. Consequently on Σ it is sufficient to know the potentials φ_{21} and φ_{23} whilst their derivatives with respect to the conormal N are easily determined.

Knowing the potential $\varphi_2 = \varphi_{21} + \varphi_{22} + \varphi_{23}$ we can find the distribution of pressure p on the surface of the plate according to the formula

$$p = p_0 - \rho a^2\left(\frac{\partial\varphi_1}{\partial\tau} + \frac{\partial\varphi_2}{\partial\tau}\right) \tag{3.15}$$

The dividing line of the surfaces Σ_1 and Σ_2 (Fig. 5) is easily determined from the two equations (3.12), (3.13), if we set $\varphi_{21} = \varphi_{23} = 0$.

Note. It was demonstrated above that the problem of diffraction of a shock wave by a contour of arbitrary shape is equivalent to the exterior problem of flow past a corresponding cylinder by a supersonic ($M = \sqrt{2}$) steady stream of ideal gas.

It can be shown also that the internal problem of flow past the cylinder corresponds to the problem of propagation of a disturbance (Fig. 1) inside a contour C, for which at large values of τ we can construct an asymptotic solution. For $\varphi(r, \theta, \tau)$ it is not difficult to obtain [6] for large values of τ

$$\varphi(r, \theta, \tau) = \frac{\alpha_0 \sin\theta}{2\pi i} \int_M \frac{I_1(rq)}{I_1'(q)} e^{q\tau} \frac{dq}{q^2} \tag{3.16}$$

For the pressure p at large values of τ we have on the contour

$$p - p_0 = -2\alpha_0 \sin\theta \sum_{k=1}^{\infty} \frac{\alpha_k}{\alpha_k^2 - 1} \sin(\alpha_k \tau) \tag{3.17}$$

where α_k are the positive zeros of the function $I_1'(q)$.

The diffraction of weak shock waves by a cascade of profiles also can be studied by the method described above.

4. Formulation of the general problem of diffraction in the case of a deformable contour. We shall assume that the relation between the pressure p on the contour C and the dimensionless deformation $\varepsilon(s, \tau)$ of the contour is given

$$p(s, \tau) = \Psi[\varepsilon(s, \tau)] \tag{4.1}$$

Here τ is the dimensionless time, s is arc length along the contour C.

First of all let us consider the case of the following relation (k is a coefficient of proportionality)

$$p(s, \tau) = k^{-1}\varepsilon(s, \tau) \tag{4.2}$$

By virtue of the fact that $p(s, \tau) \sim \partial\varphi/\partial\tau$ (where φ is the perturbation velocity potential) we have

$$\varepsilon(s, \tau) = \lambda \frac{\partial\varphi}{\partial\tau} \qquad (\lambda = -\rho a^2 k) \tag{4.3}$$

Moreover, we shall assume that the displacement ε and its derivative with respect to τ are small. Then it can be shown, just as in the case of a nondeformable contour (Section 1), that the problem of diffraction of a weak shock wave (Fig. 1) round an arbitrary deformable contour C reduces to the determination of the perturbation velocity potential $\varphi(x, y, \tau)$, satisfying the wave equation (1.5) with the boundary and initial conditions

$$\frac{\partial\varphi}{\partial n} = -\alpha_0 \frac{\partial y}{\partial n} + \varepsilon_\tau'(s, \tau) \quad \text{on } C \tag{4.4}$$

$$\varphi = 0 \quad \text{at the front of the reflected wave} \tag{4.5}$$

$$\varphi = 0 \quad \text{when } \tau \leqslant 0 \tag{4.6}$$

and that the given problem, described by the system (1.5), (4.4) to (4.6), is equivalent to the problem of external flow past a corresponding semi-infinite hollow cylinder (Fig. 2) of a steady three-dimensional supersonic ($M = \sqrt{2}$) stream of ideal gas at a small angle of attack α_0. Consequently, we shall solve the given auxiliary problem of flow past a hollow cylinder. Just as in Section 1, we apply Volterra's method [3] to the system (1.5), (4.4) to (4.6). Then when $\tau > 0$ the solution of the problem reduces to determination of the potential $\varphi(s, \tau)$ at the surface of the corresponding cylinder, satisfying the integro-differential equation

$$\varphi(s_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial\tau_0}\left\{\iint_\Sigma \left[\varphi(s, \tau)\frac{\partial V}{\partial n} - \lambda\frac{\partial^2\varphi}{\partial\tau^2}V\right] ds\, d\tau + \alpha_0 \iint_\Sigma \frac{\partial y}{\partial n} V\, ds\, d\tau\right\} \tag{4.7}$$

Here Σ is the part of the surface of the cylinder cut off by the cone of influence from the point (s_0, τ_0). Moreover, the surface of the cylinder is taken to correspond to the undeformed contour C by virtue of the smallness of ε. V is Volterra's function and has the form (3.15). Having determined the potential φ, we can calculate the remaining parameters of the flow. For example, the pressure p on the contour C is given by

$$p = p_0 - \rho a^2 (\partial\varphi / \partial\tau)C \tag{4.8}$$

Hence, substituting (4.8) in (4.2), we find that

$$\varepsilon(s, \tau) = -\lambda (\partial\varphi / \partial\tau)C \tag{4.9}$$

5. Diffraction of a shock wave in the case of a deformable circle. 1. In polar coordinates r, θ the system (1.4) to (1.6) takes the form

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = \frac{\partial^2\varphi}{\partial r^2} \tag{5.1}$$

$$\frac{\partial\varphi}{\partial r} = -\alpha_0 \sin\theta + \lambda\frac{\partial^2\varphi}{\partial\tau^2} \quad \text{when } r = 1 \tag{5.2}$$

$$\varphi = 0 \quad \text{on } ABD, \qquad \varphi = 0 \quad \text{when } \tau \leqslant 0 \tag{5.3}$$

By virtue of Section 1 the given problem is also equivalent to the problem of flow past a corresponding truncated semi-infinite hollow cylinder $r = 1 + \varepsilon(\theta, \tau)$ (Fig. 3) of a steady three-dimensional supersonic ($M = \sqrt{2}$) stream of ideal gas at a small angle $\alpha_0 = \Delta p/\rho a^2$.

As in the case of arbitrary contour C the solution of the given problem can be reduced by Volterra's method to the solution of an integro-differential equation of type (4.7), where Σ is the part of the surface of the corresponding cylinder $r = 1$, cut off by the cone of influence from the point ($r_0 = 1$, θ_0, τ_0).

An approximate expression for $\varepsilon(\theta, \tau)$ is given in [7].

For large values of τ we can construct an asymptotic solution of the problem (5.1) to (5.3). In fact, for large values of τ the radius of the whole wave surface of the flow past the cylinder depends weakly upon the angle θ and for large τ we can assume that

$$\varphi(r, \theta, \tau) = -\alpha_0 \sin\theta \, f_1(r, \tau) \tag{5.4}$$

Let us apply the Laplace transformation to the system (5.1) to (5.3) with conditions (5.4). Then the problem reduces to solution of Bessel's equation for the transformed function $F_1(r, a)$

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \left(q^2 + \frac{1}{r^2}\right)F = 0, \qquad F(r, q) = \int_0^\infty f_1(r, \tau)\, e^{-q\tau} d\tau \tag{5.5}$$

where q is the variable of the Laplace transformation, under the conditions that

$$\frac{dF}{dr} = \frac{1}{q} - \lambda q^2 F \quad \text{when } r = 1 \tag{5.6}$$

and that its derivatives with respect to τ vanish when $\tau < 0$. It is not difficult to see that the solution of the system (5.5) to (5.6) has the form

$$F(r, q) = \frac{K_1(rq)}{K_1'(q) + \lambda q K_1(q)}\, \frac{1}{q^2} \tag{5.7}$$

Consequently

$$f_1(r, \tau) = \frac{1}{2\pi i}\int_M \frac{K_1(rq)\, e^{q\tau}}{K_1'(q) + \lambda q K_1(q)}\, \frac{dq}{q^2} \tag{5.8}$$

and for large τ the potential $\varphi(r, \theta, \tau)$ is determined by formula (5.4).

Setting in (5.7) $\lambda = 0$, we obtain the solution at large τ for the rigid contour, constructed in Section 1. From (5.4) at large τ the pressure is given by

$$p = p_0 - \frac{\alpha_0 \rho a^2}{2\pi i} \sin\theta \int_M qF(1,q)\, e^{q\tau} dq \quad \text{when } r = 1 \tag{5.9}$$

or

$$p = p_0 - \alpha_0 \rho a^2 \sin\theta \Big\{ e^{-q_1(\lambda)\tau} [A(\lambda) \cos\{q_2(\lambda)\tau\} + B(\lambda) \sin\{q_2(\lambda)\tau\}] - \int_L qF(1,q)\, e^{q\tau} dq \Big\} \tag{5.10}$$

Here $-q_1 + iq_2$ is the root of the function $F^{-1}(1, q)$, $A(\lambda)$ and $B(\lambda)$ are certain completely determined functions of λ and when $\lambda = 0$ $q_1 = 0.6435$, $q_2 = 0.5012$, $A = 1.2120$, $B = 0.1898$.

In (5.10) the contour of integration L consists of the upper and lower sides of the cut along the negative axis in the q-plane, including a small circle round the point $q = 0$. Breaking the contour L into the three parts mentioned, integrating the second term in (5.10) along these paths and reckoning that

$$K_1(qe^{\pm i\pi}) = -K_1(q) \mp i\pi I_1(q), \qquad K_0(qe^{\pm i\pi}) = +K_0(q) \pm i\pi I_0(q)$$
$$K_1'(q) = -K_0(q) - K_1(q)/q, \qquad K_0(q) I_1(q) + K_1(q) I_0(q) = 1/q$$

we obtain

$$\int_L qF(1,q)\, e^{q\tau} dq = \int_0^\infty \frac{e^{-q\tau} dq}{\{[K_1'(q) + \lambda q K_1(q)]^2 + \pi^2 [I_1'(q) + \lambda q I_1(q)]^2\} q^2}$$

Substituting (5.10) in (4.1), we find that

$$\varepsilon(0, \tau) = \frac{\lambda}{2\pi i} \sin\theta \int_M qF(1,q)\, e^{q\tau} dq \tag{5.11}$$

or

$$\varepsilon(\theta, \tau) = \lambda \sin\theta \Big\{ e^{-q_1(\lambda)\tau} [A(\lambda) \cos\{q_2(\lambda)\tau\} + B(\lambda) \sin\{q_2(\lambda)\tau\}] - \int_L qF(1,q)\, e^{q\tau} dq \Big\}$$

2. Let us generalise the foregoing results to the case when

$$\Delta p = \Delta p_0 Q(\xi) \tag{5.12}$$

i.e. when the pressure difference Δp behind the front of the shock wave is a given function of the dimensionless distance ξ.

Taking advantage of the linearity of the problem, let us apply the principle of superposition, constructing a solution for $Q(\xi) = \delta(\xi)$, where $\delta(\xi)$ is the Dirac function, i.e. in the case of an extremely short incident wave.

The solution, considered in sub-section 1, was constructed for the step function $Q(\xi)$. But since p, and consequently ε also, undergo discontinuities at $\tau = 0$, it is easily seen that when $Q(\xi) = \delta(\xi)$

$$\varepsilon_\delta(\theta, \tau) = \varepsilon\underset{\tau\to 0}{(\theta, \tau)}\,\delta(\tau) + \frac{\partial\varepsilon(\theta, \tau)}{\partial\tau}$$

where $\varepsilon(\theta, \tau)$ is the solution for the step function $Q(\xi)$. Hence it follows that in the case of an arbitrary function $Q(\xi)$

$$\varepsilon_Q(\theta, \tau) = Q(0)\,\varepsilon(\theta, \tau) + \int_0^\tau Q_\xi'(\tau - \xi)\,\varepsilon(\theta, \xi)\,d\xi \tag{5.13}$$

For example, when $Q(\xi) = 1 - \mu\xi$

$$\varepsilon_\mu(\theta, \tau) = \varepsilon(\theta, \tau) - \mu\int_0^\tau \varepsilon(\theta, \xi)\,d\xi \tag{5.14}$$

where $\varepsilon(\theta, \tau)$ is determined from equations (4.7) to (4.9) or for large τ from equation (5.11).

6. Solution of the problem for a circle, when the dependence of the displacement of the contour upon the pressure on the contour is nonlinear. Let us give a generalisation of the results of Section 4 to the case when

$$p(\theta, \tau) = \Psi[\varepsilon(\theta, \tau)] \tag{6.1}$$

Let us consider two auxiliary problems.

1. Let the pressure at the front of the shock wave depend linearly upon the time, i.e.

$$\Delta p_\phi = 1 - \mu\tau \tag{6.2}$$

Then, using Duhamel's integral or formula (5.14), we obtain

$$\varepsilon_\mu(\theta, \tau) = \varepsilon(\theta, \tau) - \mu\int_0^\tau \varepsilon(\theta, \tau)\,d\tau \tag{6.3}$$

with condition (6.2), where $\varepsilon(\theta, \tau)$ is determined from equations (4.7), (4.9) or from (5.11).

2. Let the pressure $p = U(\tau)$ be applied suddenly to the contour C. Then the radially-symmetric displacement $\varepsilon_1(\tau)$ is obtained from the solution of the corresponding problem concerning flow past a hollow semi-infinite cylinder, the section of which is normal to the axis of τ. It is not difficult to see that

$$\varepsilon_1(\theta, \tau) = \frac{\lambda}{2\pi i} \int_M \frac{K_0(rq)\, e^{q\tau}}{K_0'(q) + \lambda q K_0(q)} \frac{dq}{q} \tag{6.4}$$

Let us pass to the case (6.1), reckoning that $(d\Psi/d\varepsilon)_{\varepsilon=0} = k^{-1}$, and let us set

$$\Psi_0(\varepsilon) = \Psi(\varepsilon) - k^{-1}\varepsilon \tag{6.5}$$

Then, following Baron [7], we obtain

$$\varepsilon_\Psi(\theta, \tau) = \varepsilon_\mu(\theta, \tau) - \int_0^\tau \frac{\partial \Psi_0[\varepsilon_\Psi(\theta, \xi)]}{\partial \xi} \xi_1(\tau - \xi)\, d\xi \tag{6.6}$$

Equation (6.6) is a nonlinear integral equation with respect to $\varepsilon_\Psi(\theta, \tau)$ and can be solved numerically for known values of ε_μ and ε_1 and a given form of the function $\Psi(\varepsilon)$.

The solution found for ε_Ψ from (6.6) can be generalised to the case (5.12). Just as in sub-section 2 of Section 2, we obtain

$$\varepsilon_{Q\Psi}(\theta, \tau) = Q(0)\, \varepsilon_\Psi + \int_0^\tau Q_\xi'(\tau - \xi)\, \varepsilon_\Psi(\theta, \xi)\, d\xi \tag{6.7}$$

Accordingly, the problem can be solved in general form for (6.1), (5.12).

Note. From the solution (5.8) for large τ it can be shown that the deformability of the contour can qualitatively change the pattern of the diffraction of weak shock waves round these contours.

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